Optimal Projective Spectra

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European Set Theory Conference; September 17, 2024

Introduction

Regarding maximal almost disjoint families $\mathcal{A} \subseteq [\omega]^\omega$, one may ask about the possible cardinality of $\mathcal{A}$, leading to the study of the set

$$\mathrm{spec}(\mathfrak{a}) := \{\kappa \mid \kappa = |\mathcal{A}| \wedge \mathcal{A} \subseteq [\omega]^\omega \text{ is an infinite mad family}\}.$$

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Controlling the spectrum

- Hechler showed in 1972 that it is consistent $2^{\aleph_0}$ is large and $\operatorname{spec}(\mathfrak{a}) = [\omega_1, 2^{\aleph_0}]$.
- Blass specifies a sufficient set of conditions on a set $\Theta \subseteq [\omega_1, 2^{\aleph_0}]$ guaranteeing that there is a ccc poset $\mathbb{P}$ forcing $\Theta = \operatorname{spec}(\mathfrak{a})$.
- Shelah and Spinas showed some of those assumptions can be removed.

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In other words, in a model of $V = L$, we have that $\{\mathfrak{a}\} = \mathrm{spec}(\mathfrak{a})$ has an optimal projective witness. The natural question is what happens in other models.

Background and Motivation

Guiding question

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- A Π^1_1-definable mad family is consistent with the following:
  - $\aleph_1 = \mathfrak{a} < \mathfrak{c}$, by the existence of Cohen-indestructible Π^1_1-mad families ;
  - $\aleph_1 < \mathfrak{b} = \mathfrak{a} = \mathfrak{c}$ (Brendle, Khomskii; [BK13]).

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- It is consistent $\aleph_1 < \mathfrak{a} < \mathfrak{c}$ and there is a Π^1_2 witness to $\mathfrak{a}$ (Fischer, Friedman, Schrittesser, Törnquist; [FFST22]);

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- The existence of a Π^1_2-tight mad family is consistent with $\mathfrak{b} = \mathfrak{a} = \mathfrak{c} = \aleph_2$ (Friedman, Zdomskyy; [FZ10]).

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Optimal Π^1_2-definitions

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These results are optimal:

- Mansfield's theorem states any Σ^1_2-set is contained in L or contains a perfect subset of nonconstructible reals.
- Raghavan ([Rag09]) showed that no tight mad family can contain a perfect subset.

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An Approach to realizing $\mathrm{spec}(\mathfrak{a}) = \{\aleph_1, \aleph_2\}$ with optimal witnesses

- Fix a Π^1_1 witness to $\mathfrak{a}$ in L which is "strong", in the sense that it remains maximal in a large class of forcing extensions;
- Generically add an optimally definable mad family of size $\aleph_2$, ensuring the forcing preserves the strong mad family.

Tight mad families

The notion of "strong" we use is the notion of a tight, or ω-mad family:

Definition (Malykhin, [Mal89])

An almost disjoint family $\mathcal{A} \subseteq [\omega]^\omega$ is *tight* if for all $\{X_i \mid i \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$, there exists $B \in \mathcal{I}(\mathcal{A})$ such that $|B \cap X_i| = \infty$ for all $i \in \omega$.

Where $\mathcal{I}(\mathcal{A})$ denotes the ideal on ω generated by $\mathcal{A}$, and $\mathcal{I}(\mathcal{A})^+$ is the dual filter.

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- Kurilic ([Kur01]) showed tight mad families are Cohen-indestructible.
- Guzmàn, Hrušák, and Téllez isolated a property sufficient for preserving tight mad families, show this property is preserved under countable support iterations.

Preserving tight mad families

Definition (Guzmàn, Hrušák, Téllez; [GHT20])

A proper notion $\mathbb{P}$ *strongly preserves the tightness of* $\mathcal{A}$ if for all $p \in \mathbb{P}$, countable $M \prec H_\theta$ such that $\mathbb{P}, \mathcal{A}, p \in M$ and all $B \in \mathcal{I}(\mathcal{A})$ such that $|B \cap Y| = \omega$ for each $Y \in \mathcal{I}(\mathcal{A})^+ \cap M$, there exists $q \leq p$ such that q is $(M, \mathbb{P})$-generic and

$$q \Vdash \forall \dot{Z} \in (\mathcal{I}(\mathcal{A})^+ \cap M[\dot{G}]) |\dot{Z} \cap B| = \omega.$$

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Lemma (Guzmàn, Hrušák, Téllez; [GHT20])

Countable support iterations of forcings which strongly preserve the tightness of $\mathcal{A}$ also preserve the tightness of $\mathcal{A}$.

Adding a generic Π^1_2-tight mad family

Theorem (Friedman, Zdomskyy; [FZ10])

It is consistent that $\mathfrak{b} = \mathfrak{a} = \aleph_2$ and there exists a Π^1_2-definable tight mad family.

- Countable support iteration of posets $\mathbb{Q}_\alpha$, for $\alpha < \omega_2$.
  - For α limit, $\mathbb{Q}_\alpha = \mathbb{K}$ adds a generic real $a_\alpha \subseteq \omega$, so that in the end $\mathcal{A} = \bigcup_{\alpha<\omega_2} a_\alpha$ is a tight mad family; the Π^1_2-definition is achieved by almost disjoint coding.
  - For α successor, $\mathbb{Q}_\alpha = \mathbb{H}$, Hechler forcing for adding a dominating real, to make $\mathfrak{b} = \aleph_2$.

Realizing $\mathrm{spec}(\mathfrak{a}) = \{\aleph_1, \aleph_2\}$, with optimal witnesses

We were able to show that the poset $\mathbb{K}$ to add the generic tight mad family does not itself add a dominating real:

Lemma (Fischer, M.)

The Friedman-Zdomskyy poset $\mathbb{K}$ preserves tight mad families.

Realizing $\operatorname{spec}(\mathfrak{a}) = \{\aleph_1, \aleph_2\}$, with optimal witnesses

We were able to show that the poset $\mathbb{K}$ to add the generic tight mad family does not itself add a dominating real:

Lemma (Fischer, M.)

The Friedman-Zdomskyy poset $\mathbb{K}$ preserves tight mad families.

Corollary (Fischer, M.)

It is consistent with $\aleph_1 = \mathfrak{a} < \mathfrak{c} = \aleph_2$ and there exists a Π^1_1-tight mad family of size $\aleph_1$ and a Π^1_2-tight mad family of size $\aleph_2$.

So that each $\kappa \in \operatorname{spec}(\mathfrak{a})$ has an optimally definable, tight witness.

A model with $\aleph_1 = \mathfrak{a} < \mathfrak{s} = \aleph_2$

Guiding Question

Can we obtain a similar result while also controlling other cardinal characteristics, for example in a model of $\mathfrak{b} < \mathfrak{s}$?

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Can we obtain a similar result while also controlling other cardinal characteristics, for example in a model of $\mathfrak{b} < \mathfrak{s}$?

Recall:

Definition

- $\mathfrak{s}$ is the smallest cardinality of a set $\mathcal{A} \subseteq [\omega]^\omega$ such that for every $a \in [\omega]^\omega$ there exists $b \in \mathcal{A}$ such that $|a \cap b| = |b \setminus a| = \omega$.
- $\mathfrak{b}$ is the smallest cardinality of a $\mathcal{B} \subseteq \omega^\omega$ such that there is no $f \in \omega^\omega$ with $f <^* g$ for all $g \in \mathcal{B}$.

- $\mathfrak{s} < \mathfrak{b}$ holds in the Hechler model.

$\aleph_1 = \mathfrak{a} < \mathfrak{s} = \aleph_2$ and optimal witnesses

Theorem (Shelah, 2005)

It is consistent with ZFC that $\aleph_1 = \mathfrak{b} < \mathfrak{s} = \aleph_2$.

$\aleph_1 = \mathfrak{a} < \mathfrak{s} = \aleph_2$ and optimal witnesses

Theorem (Shelah, 2005)

It is consistent with ZFC that $\aleph_1 = \mathfrak{b} < \mathfrak{s} = \aleph_2$.

- ω_2-length countable support iteration over a model of CH;
- The iterands $\mathbb{Q}$ are proper and "almost ω^ω -bounding", thus preserve the ground model reals as an unbounded family.

$\aleph_1 = \mathfrak{a} < \mathfrak{s} = \aleph_2$ and optimal witnesses

Our main result is:

Lemma (Fischer, M.)

Shelah's poset $\mathbb{Q}$ strongly preserves ground model tight mad families.

$\aleph_1 = \mathfrak{a} < \mathfrak{s} = \aleph_2$ and optimal witnesses

Our main result is:

Lemma (Fischer, M.)

Shelah's poset $\mathbb{Q}$ strongly preserves ground model tight mad families.

Theorem (Fischer, M.)

It is consistent with ZFC and $\aleph_1 = \mathfrak{a} < \mathfrak{s} = \aleph_2$ that there exists a Π^1_1-tight mad family of size $\aleph_1$, and a Π^1_2-tight mad family of size $\aleph_2$.

Conclusion

This work can lead to the following questions:

- Realizing multiple values of $\mathrm{spec}(\mathfrak{a})$ with optimal witnesses in models with $\mathfrak{c} \geq \aleph_3$?
- Is the construction compatible with a Δ^1_3 well-order of the reals ?
  - Similar questions have been asked in the case where $\mathfrak{a}$ is $\aleph_1$ or $\mathfrak{c}$ and $\mathfrak{a}$ has an optimal witness.
- What other types of combinatorial families (such as eventually different families or maximal cofinitary groups) are preserved by Shelah's forcing?

It's over

Thank you !

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